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SOLUTION OF NONSTATIONARY PROBLEMS OF HEAT CONDUCTION FOR CURVILINEAR REGIONS BY DIRECT CONSTRUCTION OF EIGENFUNCTIONS

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By separation of a time variable, the nonstationary problem is reduced to a problem on eigenvalues and eigenfunctions. The method of superposition of geometrically one-dimensional $F(\xi_i)$ solutions, where ξ_i are special variables, is employed to solve it. The integral superposition of the functions $F(\xi_i)$ yields the solution assumed. Fulfillment of the boundary conditions leads to the problem on eigenfunctions in the form of a generalized Fredholm integral equation of the first kind with known simple kernels. The resulting approximate solution of the nonstationary problem has the analytical form of a finite sum; it exactly satisfies the initial differential equation, the initial conditions, and the boundary conditions at the points of division of the boundary into small portions and approximately satisfies just the conditions between these points. A theorem on the possibility of multiplying together eigenfunctions which can be employed for regions of complex shape has been proved.

If boundary-value nonstationary problems are solved analytically, one uses either the Fourier method [1] or any integral transformation, or constructs the solution by approximate determination of eigenfunctions and eigenvalues by the method of Ritz [2] or Galerkin [3], or employs finite-difference methods [4]. In the case of a curvilinear shape of a solid body, all the methods enumerated are inefficient, since there are no grounds for construction of a specific fundamental system of functions, whose properties have been studied in depth and in sufficient detail by V. A. Il'in [5]. The exception is provided by isolated investigations in which one is able to solve applied problems for classical regions. Thus, in [6, 7], the convergence of spectral decomposition has been improved by the integral transformation of Fourier and Hankel with the use of the auxiliary quasistatic problem, and the exact solutions have been obtained. The approximate solutions obtained by simultaneous application of the methods of Kantorovich, Fourier, and Bubnov and Galerkin and the least-squares method have been given for the same classical regions in [8]. The method proposed below involves carrying out the following operations.

1. Replacement of the General Nonstationary Problem by Three Simpler Problems. Let us consider one boundary-value problem on heating of a solid body in the general formulation

$$u_t = a\Delta u + f(t, x, y)\ , \quad u\big|_\Gamma = \mu\,(t, x_\Gamma, y_\Gamma)\ , \quad u\big|_{t=0} = \varphi\,(x, y)\ . \tag{1}$$

Other boundary conditions will be considered below. We seek the solution for a certain curvilinear simply connected region Ω with boundary Γ. Problem (1) is usually subdivided into three auxiliary problems [1] and the solution is represented by the following sum:

$$u = M + v + w\ . \tag{2}$$

Here all three functions remain to be determined and can depend on t, x, and y; M is the boundary function, which may not satisfy the differential equation from (1) but is selected in such a manner as to be differentiable with respect to t and doubly with respect to x and y and to satisfy the boundary conditions from (1)

$$M\big|_\Gamma = \mu\,(t, x_\Gamma, y_\Gamma)\ . \tag{3}$$

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Let the function v satisfy the homogeneous heat-conduction equation, the homogeneous boundary condition, and the inhomogeneous initial condition

$$v_t = a\Delta v \,, \quad v\big|_{\Gamma} = 0 \,, \quad v\big|_{t=0} = \bar{\varphi} \,, \quad \bar{\varphi} = \varphi - M\big|_{t=0} \,, \tag{4}$$

and the function w satisfy the homogeneous equation with homogeneous boundary and initial conditions

$$w_t = a\Delta w + \bar{f} \,, \quad w\big|_{\Gamma} = 0 \,, \quad w\big|_{t=0} = 0 \,, \quad \bar{f} = f - M_t + a\Delta M \,. \tag{5}$$

There are no definite recommendations as far as finding M is concerned. It is particularly difficult to select it if the boundary Γ has a complex shape and dissimilar boundary conditions are specified on individual portions. Nonetheless, we must determine M, whereupon we can turn to solution of the second basic problem (4), which is related to the problem of finding eigenfunctions and eigenvalues and to solution of the third problem (5). After finding the eigenfunctions and eigenvalues, we can quite easily solve problem (5).

2. Separation of the Time Variable in Problem (4) and Its Reduction to the Problem on Finding Eigenfunctions and Eigenvalues. In solving problem (4), we will seek for v a particular solution without initial conditions

$$v = \exp\left(-a\lambda^2 t\right) R\,(x, y) \,, \quad v\big|_{\Gamma} = 0 \,. \tag{6}$$

Here the constant $(-\lambda^2)$ in the exponent remains to be determined and is negative. This conclusion follows from the theory on eigenfunctions and eigenvalues [5] and from physical considerations — the function v must tend to zero when $t \to \infty$. Substituting v from (6) into (4), for R we obtain the following problem:

$$\Delta R + \lambda^2 R = 0 \,, \quad R\big|_{\Gamma} = 0 \,. \tag{7}$$

Determination of λ values for which there exist nontrivial solutions of R from (7) is called the problem on finding eigenfunctions and eigenvalues. In [5], it has been proved that the solution of such a problem for the bounded curvilinear region Ω exists.

3.1. Introduction of the Variable ξ and Solution of the Problem on Finding Eigenfunctions and Eigenvalues. In the region Ω, we select, as a pole, a point O with a radius vector $\mathbf{r}_0$ in such a manner that any straight line E drawn through O intersects the boundary Γ just at two points: D^+ and D^-. Employing $\mathbf{r}_0$, we introduce a new geometric variable ξ:

$$\xi = (\mathbf{r} - \mathbf{r}_0)\,\mathbf{n} = (x - x_0)\cos\theta + (y - y_0)\sin\theta \,, \quad (x, y) \in \Omega \,, \tag{8}$$

where $\mathbf{n}$ is the unit normal to the straight line ξ = const, which makes an angle θ with the x axis, i.e., θ is a certain angular parameter. The variable ξ possesses the following properties. If $F(\xi)$ is a doubly differentiable function and Δ is the Laplace operator, then

$$\operatorname{grad} F\,(\xi) = F'\,(\xi)\,\mathbf{n} \,, \quad \Delta F\,(\xi) = F''\,(\xi) \,. \tag{9}$$

We will seek a particular solution of Eq. (7) without boundary conditions which depends just on one variable ξ, i.e., $R = F(\xi)$. With account for (9), Eq. (7) for $F(\xi)$ takes the form

$$F''\,(\xi) + \lambda^2 F\,(\xi) = 0 \,. \tag{10}$$

Its general solution is

$$F\,(\xi) = A\,(\theta)\cos\lambda\xi + B\,(\theta)\sin\lambda\xi \,, \tag{11}$$

where $A(\theta)$ and $B(\theta)$ are considered to be the unknown functions of the parameter θ. The differential equation (7) is linear and homogeneous; therefore, for it the principle of superposition in parameter θ holds, and then the solution of problem (7) can be represented in the following integral form:

$$R = \int_0^{\pi} [A(\theta)\cos\lambda\xi + B(\theta)\sin\lambda\xi]\, d\theta + \sum_{i=1}^{n} \left[A_i^* \cos\lambda\xi_i + B_i^* \sin\lambda\xi_i\right], \tag{12}$$

$$\xi_i = (\mathbf{r} - \mathbf{r}_0)\,\mathbf{n}_i \quad 0 < \theta_i < \pi\,, \quad i = 1, ..., n\,,$$

where $A(\theta)$ and $B(\theta)$ are the unknown functions summed up from the viewpoint of Lebesgue and A_i^*, B_i^*, θ_i, and n are the unknown coefficients, angles, and number of terms. To find all the enumerated unknowns we use the boundary conditions from (7)

$$\int_0^{\pi} \left[A(\theta)\cos\lambda\xi_\Gamma + B(\theta)\sin\lambda\xi_\Gamma\right] d\theta + \sum_{i=1}^{n} \left[A_i^* \cos\lambda\xi_{i\Gamma} + B_i^* \sin\lambda\xi_{i\Gamma}\right] = 0\,, \tag{13}$$

$$\xi_\Gamma = (\mathbf{r}_\Gamma - \mathbf{r}_0)\,\mathbf{n}\,, \quad \xi_{i\Gamma} = (\mathbf{r}_\Gamma - \mathbf{r}_0)\,\mathbf{n}_i\,.$$

Thus, the problem on finding eigenfunctions and eigenvalues has been reduced from (7) to solution of the homogeneous generalized Fredholm integral equation of the first kind (13).

As θ varies within $[0, \pi]$, the set of intersection points D^+ forms the boundary Γ^+, whereas the set of points D^- forms the boundary Γ^-. The entire boundary Γ will consist of two parts: Γ^+ and Γ^-. When $\theta = \pi$ the straight line E will coincide with its position for $\theta = 0$. If we continue to rotate the straight line E within $\theta \in [\pi, 2\pi]$, the boundary Γ will be traversed by points D^+ and D^- for the second time, which is unnecessary. These considerations substantiate the range of variation of θ in the integral of (12) and the finite sums.

To find the solution of Eq. (13) we subdivide the interval $[0, \pi]$ into small sectors $\Delta\theta_j$ ($j = 1, ..., m$) and represent the integrals in (12) and (13) by finite sums. Let the subdivision be so small ($m >> n$) that no more than one angle θ_i will find itself in each sector $\Delta\theta_j$. It is not known in advance in what sectors $\Delta\theta_j$ the angles θ_j from the finite sum in (12) and (13) will find themselves. To overcome this uncertainty we assume that the angles θ_i will find themselves in each small sector $\Delta\theta_j$. If, for example, the angles θ_i from the finite sum do not find themselves in any sector, the corresponding A_i^* and B_i^* will be equal to zero. Thus, the function R from (12) will approximately be represented by the sum

$$\begin{gathered} R = \sum_{j=1}^{m} (A_j \cos\lambda\xi_j + B_j \sin\lambda\xi_j)\,, \quad A_j = A(\theta_j^*)\,\Delta\theta_j + A_j^*\,, \\ \xi_j = (\mathbf{r} - \mathbf{r}_0)\,\mathbf{n}_j\,, \quad j = 1, ..., m\,, \quad B_j = B(\theta_j^*)\,\Delta\theta_j + B_j^*\,, \end{gathered} \tag{14}$$

where θ_j^* are certain average values of the angles θ_i in the sectors $\Delta\theta_j$.

The constants A_j and B_j consist of two parts. The first parts, $A(\theta_j^*)$, $\Delta\theta_j$, and $B(\theta_j^*)\Delta\theta_j$, depend on the method of subdivision, whereas the second parts, A_j^* and B_j^*, are independent of it. We employ this property for finding the quantities A_i^* and B_i^* and their number n in the sums of expressions (12) and (13). If the order of certain A_j and B_j remains constant with decrease in $\Delta\theta_j$, the corresponding A_j^* and B_j^* exist and their number is equal to the n sought. In constructing the solution, it is not necessary to seek the points of intersection D^+ and D^- of the straight lines E and the boundary. It is simpler to subdivide Γ into smaller portions irrespective of the angles θ_j and the straight lines E in such a manner that the points of subdivision at Γ are twice as many as the angles θ_j and to subsequently fulfill the boundary conditions at these points. Therefore, setting $\mathbf{r} = \mathbf{r}_k$ ($k = 1, ..., 2m$) in (14), we represent boundary conditions (13) in the form

$$\sum_{j=1}^{m} (A_j \cos\lambda\xi_{kj} + B_j \sin\lambda\xi_{kj}) = 0\,, \quad \xi_{kj} = (\mathbf{r}_k - \mathbf{r}_0)\,\mathbf{n}_j\,, \quad j = 1, ..., m\,, \quad k = 1, ..., 2m\,. \tag{15}$$

In (15), we have a linear algebraic homogeneous system of $2m$ equations for $2m$ unknown A_j and B_j. The condition of existence of the nontrivial solution of this system is the equality of its determinant to zero:

$$\Delta_{2m} = \left|\cos \lambda\xi_{kj}, \sin \lambda\xi_{kj}\right| = 0 \, . \tag{16}$$

This is precisely the sought characteristic equation for finding the spectrum of eigenvalues $\{\lambda_i\}$. In [5], it has been proved that all its roots λ_i are real and different. If we set $\lambda = \lambda_i$ in system (15), the determinant Δ_{2m} will vanish. By this we mean that one equation of system (15) becomes dependent on all the remaining equations; therefore, we can drop it. All the equations of the system are equally justified; then we drop, for example, the last equation for $k = 2m$ and obtain a contracted linear system with a determinant Δ_{2m-1}. When $\lambda = \lambda_i$ we should add the subscript i to the coefficients A_j and B_j in system (13), i.e., now these coefficients will be denoted as A_{ij} and B_{ij}. The coefficients A_{ij} and B_{ij} from the contracted system (15) are determined accurate to an arbitrary factor; therefore, we can easily use it and consider it to be 1, for example, $B_{im} = 1$. Once the last equation from (15) has been dropped, the contracted system of $(2m - 1)$ equations for $(2m - 1)$ unknown A_j and B_j will have the form

$$\sum_{j=1}^{m} A_{ij} \cos \lambda_i\xi_{kj} + \sum_{j=1}^{m-1} B_{ij} \sin \lambda_i\xi_{kj} = - \sin \lambda_i\xi_{km} \, , \quad k = 1, ..., 2m \, , \quad i = 1, ..., \infty \, . \tag{17}$$

We find A_{ij} and B_{ij} from system (17) and substitute them into the solution of (14). Taking into account that $B_{im} = 1$, for the eigenfunctions R_i we obtain

$$R_i = \sum_{j=1}^{m} \left[A_{ij} \cos \lambda_i\xi_j + B_{ij} \sin \lambda_i\xi_j\right] . \tag{18}$$

Finding the spectrum $\{R_i\}$ directly in explicit form is critical, since we can then construct the solution of problem (1) in the general case. This will be shown below, and now we consider the problem on finding eigenfunctions and eigenvalues for the region of complex shape and the case where the boundary conditions are specified piecewise. Thereafter we complete construction of the solution of the general problem (1).

3.2. Case of the Region of Complex Shape. Let the region Ω have a complex shape and let it be impossible to select one pole from which all the points of the boundary Γ would be seen. Then we subdivide the region Ω into several parts Ω_i having a simple shape. For the sake of definiteness, we will assume that Ω can be subdivided into two appropriate parts Ω_1 and Ω_2 by a certain line Γ_3. The entire boundary Γ is now divided by the line Γ_3 into parts Γ_1 and Γ_2. Let the boundary of the region Ω_1 consist of $\Gamma_1 \cup \Gamma_3$ and the boundary Ω_2 consists of $\Gamma_2 \cup \Gamma_3$, where Γ_3 is the shared portion of the boundary of the regions Ω_1 and Ω_2. The eigenfunctions R in Ω will be determined by the equalities

$$R = \begin{cases} R^{(1)} \, , & \text{if } (x, y) \in \Omega_1 \, ; \\ R^{(2)} \, , & \text{if } (x, y) \in \Omega_2 \, . \end{cases} \tag{19}$$

The functions $R(p)$ ($p = 1, 2$) are determined in Ω_p and vanish at Γ:

$$R^{(p)}\big|_{\Gamma_p} = 0 \, , \quad p = 1, 2 \, . \tag{20}$$

Furthermore, at the adjacent (contact) boundary Γ_3, the functions $R^{(1)}$ and $R^{(2)}$ and their derivations normal to Γ_2 must be continuous:

$$R^{(1)}\big|_{\Gamma_3} = R^{(2)}\big|_{\Gamma_3} \, , \quad \partial R^{(1)}/\partial n\big|_{\Gamma_3} = \partial R^{(2)}/\partial n\big|_{\Gamma_3} \, . \tag{21}$$

To specify the expressions of $R^{(p)}$ we select only one pole $\mathbf{r}_{0p}$ per region Ω_p so that any straight line E drawn through $\mathbf{r}_{0p}$ intersects the boundary $\Gamma_p \cup \Gamma_3$ just at two points. We introduce only one variable $\xi^{(p)}$ for each region Ω_p:

$$\xi^{(p)} = (\mathbf{r} - \mathbf{r}_{0p})\,\mathbf{n}\,, \quad p = 1, 2\,. \tag{22}$$

The unit normal $\mathbf{n}^{(p)}$ in (22) makes, as formerly, an angle θ with the x axis and θ varies within $[0, \pi]$. We apply only $2m_p$ dividing points with radius vectors $\mathbf{r}_i^{(p)}$, where $i = 1, ..., m_p$, $p = 1, 2$, at the boundaries Γ_p and $2m_3$ such points with radius vectors $\mathbf{r}_i^{(3)}$, where $i = 1, ..., 2m_3$, at Γ_3. Then a total of $2(m_p + m_3)$ dividing points will be applied at the entire boundary of the region Ω_p, i.e., at $\Gamma_p \cup \Gamma_3$. This means that there must be $(m_p + m_3)$ angular divisions $\theta_j^{(p)}$ in Ω_p, i.e., half as many as the points at the corresponding boundary. The angular divisions $\theta_j^{(p)}$ will be numbered by $j = 1, ..., (m_p + m_3)$, where $p = 1, 2$. Let us determine, similarly to (14), the functions $R^{(p)}$ in Ω_p by the expressions

$$R^{(p)} = \sum_{j=1}^{m_3+m_p} \left[A_j^{(p)} \cos \lambda\xi_j^{(p)} + B_j^{(p)} \sin \lambda\xi_j^{(p)}\right], \quad \xi_j^{(p)} = (\mathbf{r} - \mathbf{r}_{0p})\,\mathbf{n}_j^{(p)}\,, \quad p = 1, 2\,. \tag{23}$$

The expressions of $R^{(p)}$ from (23) contain $2(m_1 + m_2 + m_3)$ unknown constants $A_j^{(p)}$ and $B_j^{(p)}$. To find them we write boundary conditions (20) and (21) at the corresponding dividing points at the boundaries Γ_1, Γ_2, and Γ_3:

$$\begin{gathered}
\sum_{j=1}^{m_3+m_p} \left[A_j^{(p)} \cos \lambda\xi_{ij}^{(p)} + B_j^{(p)} \sin \lambda\xi_{ij}^{(p)}\right] = 0\,, \quad \xi_{ij}^{(p)} = (\mathbf{r}_i^{(p)} - \mathbf{r}_{0p})\,\mathbf{n}_j^{(p)}\,, \quad i = 1, ..., m_p\,, \\
\sum_{j=1}^{m_1+m_3} \left[A_j^{(1)} \cos \lambda\xi_{kj}^{(13)} + B_j^{(13)} \sin \lambda\xi_{kj}^{(1)}\right] = \sum_{j=1}^{m_2+m_3} \left[A_j^{(2)} \cos \lambda\xi_{kj}^{(23)} + B_j^{(2)} \sin \lambda\xi_{kj}^{(23)}\right], \\
\sum_{j=1}^{m_1+m_3} (\mathbf{n}_k^{(3)}\mathbf{n}_j^{(1)}) \left[B_j^{(1)} \cos \lambda\xi_{kj}^{(13)} - A_j^{(1)} \sin \lambda\xi_{kj}^{(13)}\right] = \sum_{j=1}^{m_1+m_3} (\mathbf{n}_k^{(3)}\mathbf{n}_j^{(2)}) \left[B_j^{(2)} \cos \lambda\xi_{kj}^{(23)} - A_j^{(2)} \sin \lambda\xi_{kj}^{(23)}\right], \\
\xi_{kj}^{(p3)} = (\mathbf{r}_k^{(3)} - \mathbf{r}_{0p})\,\mathbf{n}_j^{(p)}\,, \quad k = 1, ..., 2m_3\,, \quad p = 1, 2\,,
\end{gathered} \tag{24}$$

where $\mathbf{n}_k^{(3)}$ are the unit normals to Γ_3 as its dividing points $\mathbf{r}_k^{(3)}$, which are directed toward any of the regions Ω_1 and Ω_2. This homogeneous system of linear equations (24) contains respectively $2m_1$, $2m_2$, and $2m_3$ and additionally $2m_3$ equations for the same number of the unknown $A_j^{(p)}$ and $B_j^{(p)}$. By equating the determinant of the system to zero, we obtain the equation for the spectrum C3 of eigenvalues $\{\lambda_i\}$ and then, after dropping the last equation from system (24), we analogously find the unknown $A_j^{(p)}$ and $B_j^{(p)}$, thus determining the specific form of the functions $R_i^{(p)}$ from (23) and of the eigenfunctions R_i from (19).

3.3. Boundary Conditions Are Specified Piecewise. The method proposed for finding eigenfunctions and eigenvalues is also suitable for the case where the boundary conditions are specified piecewise. Let the Dirichlet conditions be specified on one part of the boundary Γ_1 and the Neumann conditions be specified at the remaining boundary Γ_2:

$$R\big|_{\Gamma_1} = 0\,, \quad \partial R/\partial n\big|_{\Gamma_2} = 0\,. \tag{25}$$

We will assume that the entire boundary is closed and consists of the parts Γ_1 and Γ_2. We have applied $2m_1$ dividing points with radius vectors $\mathbf{r}_k$, where $k = 1, ..., 2m_1$, at Γ_1 and $2m_2$ such points with radius vectors $\mathbf{r}_k$, where $k = (2m_1 + 1), ..., 2(m_1 + m_2)$, at Γ_2. A total of $2m = 2(m_1 + m_2)$ dividing points have been applied at Γ. The angles θ_j vary within $[0, \pi]$ and their total number is equal to $m = m_1 + m_2$.

The expression for the eigenfunctions R should be sought in the form (14) irrespective of the type of boundary conditions. Having fulfilled the first boundary condition from (25) at $2m_1$ points of the boundary Γ_1 and the second condition from (25) at $2m_2$ points of the boundary Γ_2, we obtain the following homogeneous linear system of $2(m_1 + m_2)$ equations for $2(m_1 + m_2)$ unknown A_j and B_j:

$$\sum_{j=1}^{m} (A_j \cos \lambda\xi_{ij} + B_j \sin \lambda\xi_{ij}) = 0 \,, \quad \xi_{ij} = (\mathbf{r}_i - \mathbf{r}_0)\, \mathbf{n}_j \,, \quad i = 1, ..., 2m_1 \,, \quad j = 1, ..., m \,;$$
$$\sum_{j=1}^{m} (\mathbf{n}_k^{(2)} \mathbf{n}_j)\, (B_j \cos \lambda\xi_{kj} - A_j \sin \lambda\xi_{kj}) = 0 \,, \quad \xi_{kj} = (\mathbf{r}_k - \mathbf{r}_0)\, \mathbf{n}_j \,, \quad k = (1 + 2m_1), ..., 2m \,, \tag{26}$$

where $\mathbf{n}_k^{(2)}$ are the unit internal normals to Γ_2 at the corresponding dividing points. By equating the determinant of system (26) to zero, we obtain the equation for the spectrum C3 of eigenvalues $\{\lambda_i\}$. After dropping the last equation from (26), we compute the corresponding A_{ij} and B_{ij}, where j = 1, ..., m and i = 1, ..., ∞, for each λ_j. After substitution of the resulting A_{ij} and B_{ij} into (14), we find the spectrum of eigenfunctions $\{R_i\}$.

The above examples show that eigenfunctions and eigenvalues can also be found just in the same manner in the case of multiply connected regions by subdividing them by auxiliary lines into several simply connected regions.

4. Finding the v-Solution of Problem (4). We will assume that the eigenfunctions R_i have been determined by expression (14) with the use of system (15), or system (24), or (26) depending on the specific case. In [5], it has been proved that the set of eigenfunctions $\{R_i\}$ forms a complete orthogonal system of functions in Ω, and if $f(x, y)$ belongs to the class of Sobolev and Liouville L_p^α, its Fourier series uniformly converges for $\alpha > 1/2$ and $\alpha_p > 2$, $p \geq 1$. We assume that the spectra $\{\lambda_i\}$ and $\{R_i\}$ have been found. Then, from (16), we obtain v_i:

$$v_i = \exp(-a\lambda_i^2 t)\, R_i\,(x, y) \tag{27}$$

and represent the solution of problem (4) by the sum

$$v = \sum_{i=1}^{\infty} C_i v_i \,, \tag{28}$$

where C_i are constants. Such a function v satisfies in construction the differential equation and the boundary conditions from (4). It remains to fulfill the initial condition, which takes the form

$$\sum_{i=1}^{\infty} C_i R_i = \overline{\varphi} \,. \tag{29}$$

The series (29) can be considered as the decomposition of $\overline{\varphi}$ into eigenfunctions $\{R_i\}$. If $\overline{\varphi} \in L_p^\alpha$, the series (29) in Ω uniformly converges; the decomposition coefficients C_i will be found from the formulas

$$C_i = \iint_\Omega \overline{\varphi} R_i dS / N_i \,, \quad N_i = \iint_\Omega R_i^2 dS \,. \tag{30}$$

Thus, we have found the function v and have obtained the solution of problem (4).

5. Finding the w-Solution of the Problem (5). We assume that $\overline{f} \in L_p^\alpha$ and it can be decomposed into $\{R_i\}$:

$$\overline{f} = \sum_{i=1}^{\infty} \alpha_i\,(t)\, R_i \,, \quad \alpha_i\,(t) = \iint_\Omega \overline{f} R_i dS / N_i \,. \tag{31}$$

We note that if f is independent of the time t, the decomposition coefficients α_i from (31) will also be independent of t. The solution of problem (5) will be sought in the form of decomposition into the eigenfunctions $\{R_i\}$ which have already been found:

$$w = \sum_{i=1}^{\infty} D_i(t) R_i, \quad D_i(0) = 0, \tag{32}$$

where $D_i(t)$ are the unknown decomposition coefficients satisfying the zero boundary conditions from (32). The quantity w determined in such a manner obviously satisfies the initial and boundary conditions from (5). To find $D_i(t)$ we substitute w from (32) and $\overline{f}$ from (31) into the differential equation (5):

$$\sum_{i=1}^{\infty} D_i' R_i = \sum_{i=1}^{\infty} a D_i \Delta R_i + \sum_{i=1}^{\infty} \alpha_i R_i. \tag{33}$$

Replacing ΔR_i from (7) by $-\alpha_i^2 R_i$ and equating the coefficients of R_i on the left- and right-hand sides of Eq. (33), we obtain the differential equation for $D_i(t)$:

$$D_i' + a\lambda_i^2 D_i = \alpha_i, \quad D_i(0) = 0. \tag{34}$$

Its solution has the form

$$D_i = \exp(-a\lambda_i^2 t) \int_0^t \alpha_i(\tau) \exp(a\lambda_i^2 \tau)\, d\tau. \tag{35}$$

Thus, the solution for w is represented by the expression

$$w = \sum_{i=1}^{\infty} R_i \exp(-a\lambda_i^2 t) \int_0^t \alpha_i(\tau) \exp(a\lambda_i^2 \tau)\, d\tau. \tag{36}$$

Finally, the solution of problem (1) has the form (2), where M satisfies boundary conditions (3), v is taken from (28), and w is taken from (36).

6. Organization of Numerical Calculation. In constructing the solution of u, the greatest difficulties arise when a linear system of the type (18) is solved. We show that when the number of dividing points $2m$ is fairly large, the determinant Δ_{2m-1} of this system can prove comparable to a "computer zero." For this purpose we subtract the $(k-1)$th line of this determinant from the kth line and write the result into the kth line:

$$A_{ij}(\cos\lambda_i\xi_{kj} - \cos\lambda_i\xi_{(k+1)j}) = A_{ij}[\cos(\lambda_i(\mathbf{r}_k - \mathbf{r}_0)\mathbf{n}_j) - \cos(\lambda_i(\mathbf{r}_{k+1} - \mathbf{r}_0)\mathbf{n}_j) =$$

$$= 2A_{ij}\sin\left[\frac{1}{2}\lambda_i(\mathbf{r}_{k+1} - \mathbf{r}_k)\mathbf{n}_j\right]\sin\left[\lambda_i\left(\frac{1}{2}(\mathbf{r}_{k+1} + \mathbf{r}_k) - \mathbf{r}_0\right)\mathbf{n}_j\right]. \tag{37}$$

The vector $(\mathbf{r}_{k+1} - \mathbf{r}_k)$ joins the kth dividing points at Γ and the $(k-1)$th point. Therefore, the quantity $h_{kj} = (\mathbf{r}_{k+1} - \mathbf{r}_k)\mathbf{n}_j$ is equal to the projection of the dividing step onto the straight line E drawn through the pole $\mathbf{r}_0$ at an angle θ_j to the x axis, i.e., h_{kj} is a small quantity. Consequently, all the lines in the transformed determinant Δ_{2m-1} will consist of small quantities. It follows that Δ_{2m-1} has an order of smallness of $\sim(h_{kj})^{2m-1}$, i.e., as the number of dividing points increases, the determinant Δ_{2m-1} decreases and can prove comparable to a "computer zero" for a certain m. This circumstance presents difficulties in numerical realization of the method. To overcome them we should normalize the algebraic system (15) or (24), or (26) depending on the case. For this purpose, we compute the determinant Δ_{2m-1} and then divide each equation of system (15) by the quantity $K_{2m-1} = (\Delta_{2m-1})^{1/(2m-1)}$. If Δ_{2m-1} is so small that the quantity K_{2m-1} proves to be computed with a large error, we should compute once again the determinant $\Delta^{(1)}_{2m-1}$ in the resulting new system of equations. If the determinant proves different from 1, each equation of the new system must be divided by the quantity $K_{2m-1} = (\Delta_{2m-1})^{1/(2m-1)}$ for the second time. Such a transformation must be performed until the determinant of the system of $(2m-1)$ equations becomes equal to 1. Once the process of normalization of system (15) is completed, which is usually attained in two operations, all the unknowns are easily computed.

7. Construction of Eigenfunctions for Complex Regions with Complex Boundary Conditions. Of considerable utility can prove the theorem on multiplication of eigenfunctions together. Eigenfunctions are usually employed in the form of linear combinations; here we prove a theorem on the possibility of multiplying them together in a particular case.

We introduce the definition of geometric orthogonality of any two differentiable functions $U_1(x, y, z)$ and $U_2(x, y, z)$. Their level surfaces have the form

$$U_1(x, y, z) = C_1\,, \quad U_2(x, y, z) = C_2\,. \tag{38}$$

The functions U_1 and U_2 will be called geometrically orthogonal if their level surfaces are mutually orthogonal, i.e.,

$$(\mathrm{grad}\, U_1)\,(\mathrm{grad}\, U_2) = 0\,. \tag{39}$$

We give the simplest example. If $U_1 = U_1(x)$ and $U_2 = U_2(y)$, then we have (grad U_1) ((grad U_2) = 0).

Theorem. *Let two eigenfunctions U_1 and U_2 be solutions of the problem*

$$\Delta U_p + \lambda_p^2 U_p = 0\,, \quad (x, y, z) \in \Omega_p\,, \quad L_p\,[U_p]_{\Gamma_p} = 0\,, \quad p = 1, 2\,, \tag{40}$$

where L_1 and L_2 are the linear operators corresponding to a homogeneous boundary condition. We introduce the notation $\Omega = \Omega_1 \cap \Omega_2$ and denote its boundary as $\Gamma = \Gamma_1^* + \Gamma_2^*$, where $\Gamma_1^* = \Gamma_1 \cap \Omega_2$ and $\Gamma_2^* = \Gamma_2 \cap \Omega_1$ are the parts of the boundaries Γ_1 and Γ_2 respectively. Let the regions Ω_1 and Ω_2 be such that their boundaries Γ_1 and Γ_2 are orthogonal and the two eigenfunctions U_1 and U_2 possess the property of geometric orthogonality (39). Then the product of these eigenfunctions

$$u(x, y, z) = U_1 U_2 \tag{41}$$

will also be eigenfunction in the region Ω with boundary Γ and will be the solution of the problem

$$\Delta u + \lambda^2 u = 0\,, \quad L_1\,[u]_{\Gamma_1^*} = 0\,, \quad L_2\,[u]_{\Gamma_2^*} = 0\,, \tag{42}$$

where

$$\lambda^2 = \lambda_1^2 + \lambda_2^2\,. \tag{43}$$

Proof. The Laplace operator of the function u will be represented as follows:

$$\Delta u = (\mathrm{grad})\,(\mathrm{grad}\, U_1 U_2) = (\mathrm{grad})\,(U_1\, \mathrm{grad}\, U_2 + U_2\, \mathrm{grad}\, U_1) =$$

$$= U_1 \Delta U_2 + U_2 \Delta U_1 + 2\,(\mathrm{grad}\, U_1)\,(\mathrm{grad}\, U_2) = U_1 \Delta U_2 + U_2 \Delta U_1\,. \tag{44}$$

Here the last equality has been obtained with the use of the property (39). We eliminate ΔU_p from (44) using (40):

$$\Delta u = -\lambda_2^2 U_1 U_2 - \lambda_1^2 U_2 U_1 = -(\lambda_1^2 + \lambda_2^2)\, U_1 U_2 = -\lambda^2 u\,. \tag{45}$$

It follows from (45) that $u = U_1 U_2$ satisfies Eq. (42). Let us analyze the boundary conditions for u. Since the boundaries Γ_1^* and Γ_2^* are mutually orthogonal the operator L_1 at the boundary Γ_1^* does not influence the function U_2 and, conversely, the operator L_2 at Γ_2^* does not influence U_1. Then the boundary conditions for u can be represented in the form

$$L_1\,[U_1U_2]_{\Gamma_1^*} = U_2L_1\,[U_1]_{\Gamma_1^*} = 0 \Rightarrow L_1\,[u]_{\Gamma_1^*} = 0\,,$$

$$L_2\,[U_1U_2]_{\Gamma_2^*} = U_1L_2\,[U_2]_{\Gamma_2^*} = 0 \Rightarrow L_2\,[u]_{\Gamma_2^*} = 0\,.$$

Hence we obtain the proof of the theorem, which makes it much easier to find eigenfunctions and eigenvalues for complex regions with complex boundary conditions if the corresponding conditions are fulfilled.

The results obtained can be generalized to the case of spatial problems. The simplicity and high accuracy make it possible to employ this method for solution of complicated practical engineering problems.

NOTATION

a, thermal diffusivity, m^2/sec; $A(\theta)$ and $B(\theta)$, unknown functions, K; A_i^*, B_i^*, A_i, B_i, A_j, and B_j, unknown coefficients, K; C_1 and C_2, constants, K; C_i and $D_i(t)$, coefficients of spectral decompositions of the functions v and w; D^+ and D^-, points of intersection of these rays and Γ; E and $\mathbf{n}$, rays and unit vectors along them drawn at an angle θ to the x axis; $F(\xi_i)$, one-dimensional particular solution, K; f, internal source, K/sec; $\overline{f}$, internal source in the equation for w, K/sec; h_{kj}, projection of the dividing step of the boundary, m; L_1 and L_2, operators of the boundary conditions; L_p^{α}, functional sets of Sobolev and Liouville, where α and p are the parameters of a set; M and v and w, boundary function and auxiliary functions, K; $2m$, number of dividing points of the boundary; n, unknown number of terms; $\mathbf{r}_0$, radius vector of the pole; $\mathbf{r}$, $\mathbf{r}_\Gamma$, and $\mathbf{r}_k$, radius vectors of arbitrary points in Ω, at Γ, and at the points of division of Γ into small parts respectively; $\mathbf{r}_{0p}$, radius vectors of the poles in Ω_p; R, eigenfunction, K; $R^{(1)}$ and $R^{(2)}$, components of the eigenfunction R in the regions Ω_1 and Ω_2, K; t, time, sec; U_1 and U_2, two geometrically orthogonal eigenfunctions, K; u, temperature, K; v_i, particular solutions, K; x, y, Cartesian coordinates, m; x_Γ, y_Γ, coordinates of the points at Γ, m; $\alpha_i(t)$, coefficient of spectral decomposition of the function $\overline{f}$, 1/sec; Γ_1 and Γ_2, parts of the boundary Γ; Γ_1^* and Γ_2^*, parts of the boundaries Γ_1 and Γ_2; Δ, Laplace operator; Δ_{2m}, characteristic determinant; Δ_{2m-1}, determinant of the contracted system; θ_j^*, angles in the sectors $\Delta\theta_j^*$; λ, eigenvalues, 1/m; μ and φ, boundary and initial conditions, K; ξ_i, special variables, m; ξ_Γ, $\xi_{i\Gamma}$, and ξ_{kj}, values of the variable ξ at Γ, m; $\xi^{(p)}$ and $\xi_j^{(p)}$, special variables in Ω_p for an arbitrary ray and for the ray along $\mathbf{n}_j^{(p)}$, i.e., the unit vectors, m; $\xi_{ij}^{(p)}$, special variables at the boundaries Γ_p, m; $\xi_{kj}^{(p3)}$, special variables at the dividing points of the boundary Γ_3, m; φ, initial condition for v, K; Ω and Γ, region and its boundary; Ω_1 and Ω_2, parts of the region Ω; Γ_3, their adjacent boundary. Subscripts: i, Nos. of terms in the finite sum; n, number of terms in the finite sum; j, Nos. of the dividing points of the boundary Γ; $2m$, number of dividing points; p = 1, 2, Nos. of simply connected regions.

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